Math 255A Lecture 27 Notes

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December 6, 2018

1 Reflexive Spaces and Kakutani's theorem

1.1 Helly's lemma

Let B be a Banach space. Recall that B is called reflexive if the natural map $J : B \to B^{**}$ given by $x \mapsto (\xi \mapsto \langle x, \xi \rangle)$ is a bijection. We want to characterize reflexive spaces. First we need a lemma.

Lemma 1.1 (Helly)**.** *Let B be a Banach space, and let $\xi_1, \dots \xi_m \in B^*$ and $\alpha_1, \dots, \alpha_n \in K$ ($= \mathbb{R}$ or $\mathbb{C}$). The following conditions are equivalent:*

1. *For each $\varepsilon > 0$, there exists $x_\varepsilon \in B$ such that $\|x_\varepsilon\| \leq 1$ and $|\langle x_\varepsilon, \xi_j \rangle - \alpha_j| < \varepsilon$ for $1 \leq j \leq n$.*
2. *For all $\beta_1, \dots, \beta_n \in K$, $|\sum_{j=1}^n \beta_j \alpha_j| \leq \|\sum_{j=1}^n \beta_j \xi_j\|$.*

Proof. (1) $\implies$ (2): Let $\beta_1, \dots, \beta_n \in K$ be given, and let $S = \sum_{j=1}^n |\beta_j|$. For $\varepsilon > 0$, we have

$$\left| \sum_{j-1}^n \beta_j \langle x_\varepsilon, \xi_j \rangle - \sum_{j=1}^n \beta_j \alpha_j \right| < \varepsilon S \implies \left| \sum_{j=1}^n \beta_j \alpha_j \right| \leq \left\| \sum_{j=1}^n \beta_j \xi_j \right\| \underbrace{\|x_\varepsilon\|}_{\leq 1} + \varepsilon S.$$

Letting $\varepsilon \to 0$ gives us $|\sum_{j=1}^n \beta_j \alpha_j| \leq \|\sum_{j=1}^n \beta_j \xi_j\|$.

(2) $\implies$ (1): Consider the linear, continuous map $F : B \to K^n$ sending $x \mapsto (\langle x, \xi_1 \rangle, \dots, \langle x, \xi_n \rangle)$. Then condition 1 holds if and only if $(\alpha_1, \dots, \alpha_n) \in F(\{x \in B : \|x\| \leq 1\})$. Assume that $(\alpha_1, \dots, \alpha_n) \notin F(\{x \in B : \|x\| \leq 1\})$ (which is closed and convex in K^n). By the geometric Hahn-Banach theorem, there exists a continuous, linear form f on K^n and $\gamma \in \mathbb{R}$ such that $\mathrm{Re}(f(y)) < \gamma < \mathrm{Re}(f(\alpha_1, \dots, \alpha_n))$ for all $y \in F(\{x \in B : \|x\| \leq 1\})$. Writing $f(y) = \beta \cdot y = \sum_{j=1}^n \beta_j y_j$, we get

$$\mathrm{Re}\left(\sum_{j=1}^n \beta_j \langle x, \xi_j \rangle \right) < \gamma < \mathrm{Re}\left(\sum_{j=1}^n \alpha_j \beta_j \right) \leq \left| \sum_{j=1}^n \alpha_j \beta_j \right|$$

for all $x \in B$ with $\|x\| \leq 1$. So

$$\operatorname{Re}\left\langle x, \sum_{j=1}^{n} \beta_j \xi_j \right\rangle < \gamma < \left| \sum_{j=1}^{n} \alpha_j \beta_j \right|$$

for all $x \in B$ with $\|x\| \leq 1$. Replacing x by $e^{i\theta} x$ (with $\theta \in \mathbb{R}$), we get by varying θ that

$$\left\langle x, \sum_{j=1}^{n} \beta_j \xi_j \right\rangle < \gamma < \left| \sum_{j=1}^{n} \alpha_j \beta_j \right|$$

if $\|x\| \leq 1$. That is,

$$\left\| \sum_{j=1}^{n} \beta_j \xi_j \right\| \leq \gamma < \left| \sum_{j=1}^{n} \alpha_j \beta_j \right|,$$

which contradicts condition 2. $\square$

Lemma 1.2. *Let B be a Banach space. Then the set $J(\{x \in B : \|x\| \leq 1\})$ is dense in $\{z \in B^{**} : \|z\| \leq 1\}$ for the weak topology $\sigma(B^{**}, B^*)$.*

Proof. Let $z \in B^{**}$ with $\|z\| \leq 1$, and let V be an open neighborhood of z in the topology $\sigma(B^{**}, B^*)$. We claim that $V \cap J(\{x \in B : \|x\| \leq 1\}) \neq \varnothing$. We may assume that V has the form $V = \{y \in B^{**} : |\langle \xi_j, y - z \rangle| < \varepsilon \; \forall 1 \leq j \leq n\}$ with $\varepsilon > 0$ and $\xi_j \in B^*$. We must show that there exists $x \in B$ with $\|x\| \leq 1$ such that $|\langle x, \xi_j \rangle - \langle \xi_j, z \rangle| < \varepsilon$ for $1 \leq j \leq n$. Letting $\alpha_j = \langle \xi_j, z \rangle$, we notice that for all $\beta_1, \dots, \beta_n \in K$,

$$\left| \sum_{j=1}^{n} \beta_j \alpha_j \right| = \left| \left\langle \sum_{j=1}^{n} \beta_j \xi_j, z \right\rangle \right| \leq \left\| \sum_{j=1}^{n} \beta_j \xi_j \right\|_{B^*} \underbrace{\|z\|_{B^{**}}}_{\leq 1} \leq \left\| \sum_{j=1}^{n} \beta_j \xi_j \right\|_{B^*}.$$

By the previous lemma, there exists an $x_\varepsilon \in B$ with $\|x_\varepsilon\| \leq 1$ such that $|\langle x_\varepsilon, \xi_j \rangle - \alpha_j| < \varepsilon$. Thus, $J(x_\varepsilon) \in J(\{x \in B : \|x\| \leq 1\}) \cap V$. $\square$

Remark 1.1. Notice that $J(\{x \in B : \|x\| \leq 1\}) \subseteq \{z \in B^{**} : \|z\| \leq 1\}$ is closed in the strong sense.

1.2 Kakutani's theorem

Proposition 1.1. *Let B_1, B_2 be Banach spaces, and let $T \in \mathcal{L}(B_1, B_2)$. Then $T : (B_1, \sigma(B_1, B_1^*)) \to (B_2, \sigma(B_2, B_2^*))$ is continuous.*

Proof. Let $O \subseteq B_2$ be open for $\sigma(B_2, B_2^*)$. We may assume that $O = \{y \in B_2 : |\langle y - x, \eta_j \rangle| < \varepsilon \,\forall 1 \leq j \leq n\}$, where $x \in B_2$, $\eta_j \in B_2^*$, and $\varepsilon > 0$. Then

$$\begin{aligned} T^{-1}(O) &= \{z \in B_1 : |\langle Tz - x, \eta_j \rangle| < \varepsilon \,\forall 1 \leq j \leq n\} \\ &= \{z \in B_1 : |\langle Tz, \eta_j \rangle - \langle x, \eta_j \rangle| < \varepsilon \,\forall 1 \leq j \leq n\} \\ &= \{x \in B_1 : |\langle z, T^*\eta_j \rangle - \langle x, \eta_j \rangle| < \varepsilon \,\forall 1 \leq j \leq n\}, \end{aligned}$$

which is open in B_1 for $\sigma(B_1, B_1^*)$ since $T^*\eta_j \in B_1^*$. □

Theorem 1.1 (Kakutani). *A Banach space B is reflexive if and only if the closed unit ball $\{x \in B : \|x\| \leq 1\}$ is compact for the weak topology $\sigma(B, B^*)$.*

Proof. Assume first that B is reflexive. Then $J(\{x \in B : \|x\|_B \leq 1\}) = \{y \in B^{**} : \|y\|_{B^{**}} \leq 1\}$ is compact in the weak* topology $\sigma(B^{**}, B^*)$ by Banach Alaoglu. We only have to check that $J^{-1} : (B^{**}, \sigma(B^{**}, B^*)) \to (B, \sigma(B, B^*))$ is continuous (as a continuous image of a compact set is compact). When $O = \{y \in B : |\langle y - x, \xi \rangle| < \varepsilon\}$ with $x \in B$, $\xi \in B^*$ and $\varepsilon > 0$ is open in $\sigma(B, B^*)$, it suffices to check that $(J^{-1})^{-1}(O) = J(O)$ is open in B^{**} with respect to $\sigma(B^{**}, B^*)$. This follows from $J(O) = \{z \in B^{**} : |\langle \xi, z \rangle - \langle x, \xi \rangle| < \varepsilon\}$.

Assume that $\{x \in B : \|x\| \leq 1\}$ is compact for the weak topology $\sigma(B, B^*)$. We claim that the map $J : (B, \sigma(B, B^*)) \to (B^{**}, \sigma(B^{**}, B^{***}))$ is continuous. Indeed, $J : B \to B$ is strongly continuous (as an isometry), and so the claim follows. Now $B^* \subseteq B^{***}$, so the topology $\sigma(B^{**}, B^*)$ on B^{**} is weaker than $\sigma(B^{**}, B^{***})$, and it follows that $J : (B, \sigma(B, B^*)) \to (B^{**}, \sigma(B^{**}, B^*))$ is continuous. We get that $J(\{x \in B : \|x\| \leq 1\})$ is compact for $\sigma(B^{**}, B^*)$ as the continuous image of a compact set is compact. It is also dense in $\{x \in B^{**} : \|z\| \leq 1\}$ for the topology $\sigma(B^{**}, B^*)$. Therefore, $J(\{x \in B : \|x\| \leq 1\}) = \{z \in B^{**} : \|z\| \leq 1\}$ and hence, $J(B) = B^{**}$. So B is reflexive. □